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Möbius Geometry and Cyclidic Nets: A Framework for Complex Shape Generation

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Abstract

Free-form architecture challenges architects, engineers and builders. The geometrical rationalization of complex structures requires sophisticated tools. To this day, two frameworks are commonly used: NURBS modeling and mesh-based approaches. The authors propose an alternative modeling framework called *generalized cyclidic nets* that automatically yields optimal geometrical properties for the façade and the structure. This framework uses a base circular mesh and Dupin cyclides, which are natural objects of the geometry of circles in space, also known as Möbius geometry. This paper illustrates how new shapes can be generated from generalized cyclidic nets. Finally, it is demonstrated that this framework gives a simple method to generate curved-creases on free-forms. These findings open new perspectives for structural design of complex shells.

Keywords: conceptual design, structural morphology, architectural geometry, fabrication-aware design, curved crease

1. Introduction

Non-standard architecture often makes reference to complex doubly-curved systems. Bagneris *et al.* [2] identify three design approaches for non-standard architecture: geometrically-constrained forms or “analytic forms”, mechanically-constrained forms or “mechanical forms” and “flexible forms”. Geometrically-constrained strategy uses compositions of geometries which are known by the builder, linking form and construction constraints. Mechanical forms are shapes that are mechanically optimal with respect to certain load cases; the form is linked with structural performance. Flexible forms consider other aspects of an architectural project: shape is not thought with respect to construction nor structure, but to other considerations. The first approach is present since the beginning of architecture. The two latter appeared during the twentieth century due to innovation in computation, like force-density for mechanical forms (Scheck [20]) or Bézier surfaces or De Casteljau’s algorithm for flexible approaches.

Flexible forms are now widely used, implying heavy post-rationalization techniques and a workflow where the architects and structural engineers need feedback from geometry specialists. This kind of approach has severe limitations in terms of time and budget during the early stages of the design. This fact triggered new reflection on *geometrically-constrained* shape explorations (Yang *et al.* [22], Deng *et al.* [5]). These methods are based on functional minimization written on a mesh. Common optimization targets include flat panels (Glymph *et al.* [8]) and torsion-free nodes (Liu *et al.* [13]). These methods however suffer from some limitations: change of topology (for example from quads to hexagons) is difficult and the combination of torsion-free nodes with flat quad panels is only possible when the lines of the meshes follow lines of curvature of the underlying surface, which means that the initialization of the problem is crucial for the convergence of the algorithm (Liu *et al.* [13]). A unified framework that could allow interactive remeshing and intuitive design of Planar Quadrilateral (PQ)-Meshes with torsion-free nodes would therefore be a step towards a more efficient geometrically-constrained design approach.

Furthermore, most recent research link discrete differential geometry with notions of smooth geometry (Liu *et al.* [13]), excluding hence surfaces with discontinuity of normal vectors, as some notions like curvature cannot be defined everywhere. Such surfaces constitute however an important family of surfaces for the designers (think of folded structures). The topic of doubly curved creases in shell structures needs new methods and new tools to take advantage of their potential.

This paper presents new tools for geometrically-constrained design strategy. Main contributions include:

- a new framework for complex shape generation allowing automatic discretization with torsion-free nodes;
- implementation of this framework and application to some archetypal examples;
- a new insight on doubly-curved crease which opens new formal possibilities for shells and spatial structures.

The paper is organized as follows: Section 2 introduces key geometrical notions used in the framework introduced by the authors. Section 3 proposes the implementation of the new framework based on *cyclidic nets* and its application. In the fourth Section, the authors generalize the notion of cyclidic nets to surfaces with creases. The structural potential offered by this theoretical finding is illustrated on Section 5. A brief conclusion sums up the findings of this paper and proposes some developments to this work.

2. Cyclidic Nets and Möbius Geometry

2.1 General geometrical definitions

2.1.1. Conical Mesh

When building a structural layout, the orientation of members is an important geometrical problem to be solved. A geometry where every node can be given an axis is of interest. Such nodes are called *torsion free-nodes*. Trivial solutions for this problem exist (like translation along a constant vector), but non-trivial solution for offsets with *torsion-free nodes* requires the notion of *Conical Meshes* studied in (Liu *et al.* [13]). It was also proven in (Liu *et al.* [13]) that the only quadrilateral meshes that are planar and conical follow the lines of curvature of the underlying surfaces, which makes the computation of lines of curvature a key-issue for structural designers. The case studies published in the aforementioned paper suggest that Conical Meshes can be found on any surface.

2.1.2. Edge Offset Meshes

Edge Offset Meshes are a subclass of Conical Meshes. They correspond to the case where all beams axes have a constant angle with the node normal. The consequence is that beams of constant height are perfectly aligned on top and bottom of the node axis, hence the name *perfect node* sometimes found in literature (Pottmann *et al.* [18]). Such meshes are only possible on *isothermic surfaces*, a restricted family of surfaces; among them: surfaces of revolution, moulding surfaces (Mesnil *et al.* [15, 16]), minimal and constant-mean curvature surfaces. Interactive modeling of surfaces with perfect nodes is still a challenge for designers and researchers.

2.1.3. Circular Meshes

A *Circular Mesh* is a quadrilateral mesh where each face is inscribed in a circle. They are related to PQ-Conical Meshes by a form of duality described in (Pottman and Wallner [19]). It is indeed possible to convert any circular mesh into a quadrilateral conical mesh, and vice versa. Like Conical Meshes, Circular Meshes are seen as a discrete parameterization of surfaces by lines of curvature.

2.1.4. Möbius Transform

Circles are the natural shapes describing Circular Meshes. It is legitimate to study transformations that map circles to circles because such transformations preserve the circular property in meshes. They are known as Möbius transforms, or more commonly inversions with respect to a sphere. The geometry of circles in space is called *Möbius Geometry*.

An inversion is defined by a center C, and a ratio k. The image of a point M is given by:

$$\begin{cases} \overline{CM}.\overline{CM'} = k \\ C, M, M' \, aligned \end{cases} \tag{1}$$

It is interesting to notice that Möbius transforms do not preserve the global shape. These transforms give therefore a way to apply global deformations to meshes while preserving local properties. The potential of these simple transformations is detailed in Section 3.2.

2.2. Cyclidic Meshes

2.2.1. Dupin Cyclide

This work uses a key object in discrete differential geometry: Dupin cyclides. These surfaces were discovered by the French mathematician Charles Dupin, who studied some of their remarkable properties in 1803. Dupin cyclides can be defined as inversion of tori in the sense of Section 2.1.4. Some special cases of Dupin cyclides include tori, cylinders and spheres. An example of a cyclide with patches following the lines of curvature is shown on Figure 1.

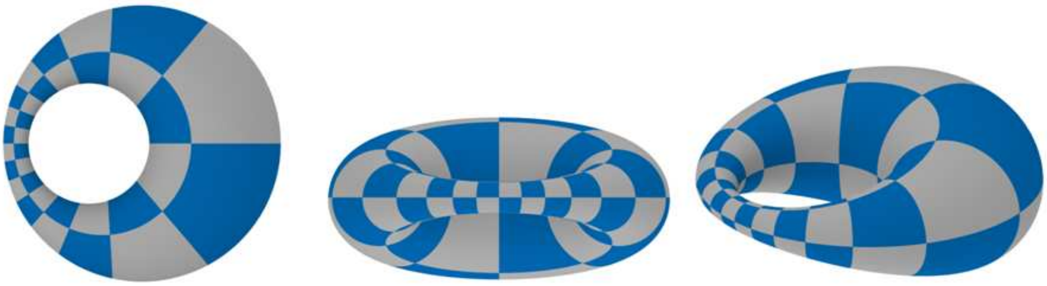

Figure 1: Dupin Cyclide and quads delimited by lines of curvature (top, front and perspective)

For application in architecture, some properties of cyclides are particularly appealing:

- their lines of curvature are circles;
- a quad whose edges are lines of curvature is inscribed in a circle: lines of curvatures thus create natural *Circular Meshes;*
- they are isothermic surfaces (Adam [1]) and can therefore be covered with Edge Offset Meshes (Pottmann [18]).

Dupin cyclides are also easily parameterized by lines of curvature, as seen in Section 2.1.2. This guarantees good properties for other meshes than quad meshes, since planar hexagonal meshes follow lines of curvature (Wang *et al.* [21]).

2.1.2. Cyclidic Patch

The four intersections of lines of curvature in cyclides naturally define a circle. Conversely, a cyclic quadrilateral and a frame give a unique portion of cyclide, later called *cyclidic patch* in this paper. Several algorithms have been proposed to convert a cyclidic patch to a NURBS surface, the one used in this paper has been proposed in Garnier *et al.* [7]. The algorithm requires a cyclic quadrilateral and an orthogonal frame (one blue and one red arrow on Figure 2). The other frames are generated by reflection with respect to the median plane of each edge of the quad. Note that the median planes are congruent on Figure 2. The boundaries of the patch are circles which are uniquely defined by two points and two (compatible) tangent vectors.

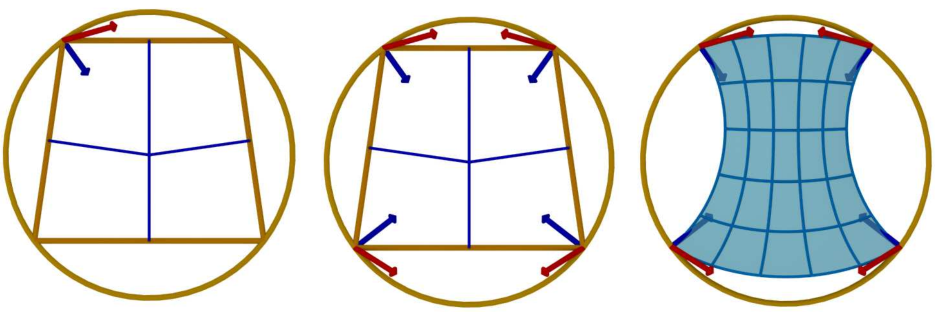

Figure 2: Generation of a cyclidic patch from a cyclic quadrilateral and one frame

The resulting surface is naturally parameterized by its lines of curvatures. Since the underlying surface is a Dupin cyclide, this means that the trivial quad meshes on cyclidic patches like the one displayed in Figure 2 are exact circular meshes.

2.1.3. Cyclidic Nets

The properties of cyclides and the existence of a conversion algorithm to NURBS led to the idea of representing shapes as a collection of cyclidic patches. The mathematical properties of such shapes, called cyclidic nets, have been studied in (Huhnen-Venedey and Bobenko [9]). Cyclidic nets are based on Circular Quadrilateral Meshes and require only one frame vector, the others being generated by reflection if they belong to the same cyclidic patch. A simple reflection rule illustrated in Figure 5 allows the propagation of the frame to adjacent patches.

A strong limitation of these objects is that any vertex of the Circular Mesh should have a valence of four. This was seen as a problem for the modeling of shapes with umbilical points. However, recent advances show that this limitation has been solved and that all shapes can be approximated by cyclidic nets (Krasauskas [12]). Starting from circular boundary curves (in red on Figure 3), it is possible to generate a circular mesh that supports a cyclidic net aligned with the boundaries. This process shrinks the opening, and a finite number of iterations can make the opening arbitrary small (for example the size of a panel).

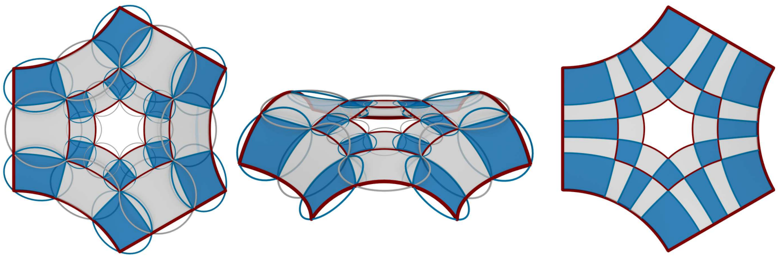

Figure 3: Filling of a n-sided hole with circular boundaries: circular mesh created by the method of Krasauskas [12] and resulting cyclidic net (right)

Applications of cyclidic nets for architecture were proposed in (Bo *et al.* [4]), the cyclidic patches being typically the size of panels. This solution has the advantage of generating one unique type of node, but it requires building with circular arcs. This solution would also require remeshing if the designer wants to modify the structural layout. The opportunity offered by larger cyclidic patches as global modeling tool is unexplored up to now.

3. Shape generation framework

3.1. Cyclidic Nets Framework

3.1.1. Framework

The study of cyclidic nets demonstrates that a circular mesh and a frame can generate a collection of NURBS parameterized by their lines of curvature. This collection of surfaces can then be easily meshed with conical or circular PQ-Meshes or planar hexagonal meshes. Unlike previous applications of cyclidic meshes in architecture, it is suggested by the authors that the shapes can be described with large cyclidic patches. The result is a framework for shape modeling tailored for architectural constraints.

A brief comparison with NURBS is proposed in Table 1. A parallel can be found between the control polygon of NURBS and the circular mesh of cyclidic nets. Unlike control points in NURBS modeling, the vertices of Cyclidic Nets are all on the modeled surface. The surface resulting from cyclidic nets are only C^1, which is a drawback in many industries, but is not a very serious issue in architectural design. Indeed, the final shape is very often built with flat or developable panels, which makes the built envelope at most a C^1 surface. Finally, both NURBS modeling and Cyclidic Nets encounter difficulties when modeling complex topologies. This led to alternative modeling techniques, like surfaces of subdivision in the continuity of NURBS (Liu *et al.* [13]), and hole-filling strategies like recalled in this paper for cyclidic nets.

	NURBS	**Cyclidic Net**
Base shape	Control Polygon	Circular Quad Mesh + one frame
Interpolation	Bernstein polynomial	Cyclidic Patch
Surface regularity	From C^0 to C^∞	From C^0 to C^1
Isoparametric lines properties	None	Curvature lines
Complex topologies	T-splines, subdivision surfaces	Hole filling

Table 1: Comparison of NURBS and Cyclidic Net in architecture

3.1.2. Primitives for circular meshes

The framework proposed here requires thus circular meshes as input. Some shapes give trivial conical or circular meshes. Among them, surfaces of revolution, moulding surfaces or Monge surfaces (Mesnil *et al.* [15,16]). It is possible to apply Möbius transforms to those shapes to enrich the formal vocabulary: some case studies are presented in this paper. Composition of these shapes is also possible in the manner of what has been proposed for scale-trans surfaces (Glymph *et al.* [8]).

This framework can be combined with the optimization methods described in (Deng et al. [4]) and (Yang et al. [13]), with an optimization of quad meshes towards circular meshes. The combination of those methods has several advantages. First of all, since cyclidic patches are smooth surfaces, only a few of them are required to describe a given shape: this decreases the size of the problems to be solved by optimization and makes shape exploration easier. Secondly, they can be easily remeshed with no computational optimization.

3.1.3. Implementation and numerical issues

The framework proposed in this paper has been implemented within Grasshopper. It allows a fully parametric design approach and interaction with other Grasshopper plug-ins such as Karamba 3d. The geometrical tools generate the cyclidic nets and the associated subdivisions. Once the circular mesh is chosen, an infinity of frames can be chosen. All the underlying surfaces are C^1, but some are visually more pleasant than others. To take this aesthetic aspect into account, a fairness-functional has been introduced to give the smoothest possible shape for a given circular mesh. The fairness function to minimize is here defined by:

$$F(\theta, \lambda) = \sum_{edges} \int \kappa^2 ds \tag{2}$$

where κ is the curvature of each edge and (θ, λ) represent the spherical coordinates of the first frame vector. The functional recalls therefore a bending energy: it can be minimized by classical methods with respect to (θ, λ). The optimization is here done using the BFGS algorithm, a classical quasi-Newton scheme. The initial frame is chosen at a boundary and lay in the face plane. The functional varies of less than 0.1% in less than 5 steps, which demonstrates fast convergence.

Other fairness functions based on the variation of curvature or Willmore energy of the underlying surfaces could be used (Joshi and Séquin [10]). From a technical point of view, their computation would be very efficient because it is possible to retrieve the implicit equation of the cyclide for a patch (Garnier *et al.* [7]). The minimization scheme would just have to be adapted to the new functionals.

3.2. Applications

Möbius transforms applied on circular meshes offer a rich variety of shapes to support cyclidic nets. An example is given on Figure 4: a simple surface of revolution is inverted to give a less obvious “peanut-shaped” geometry.

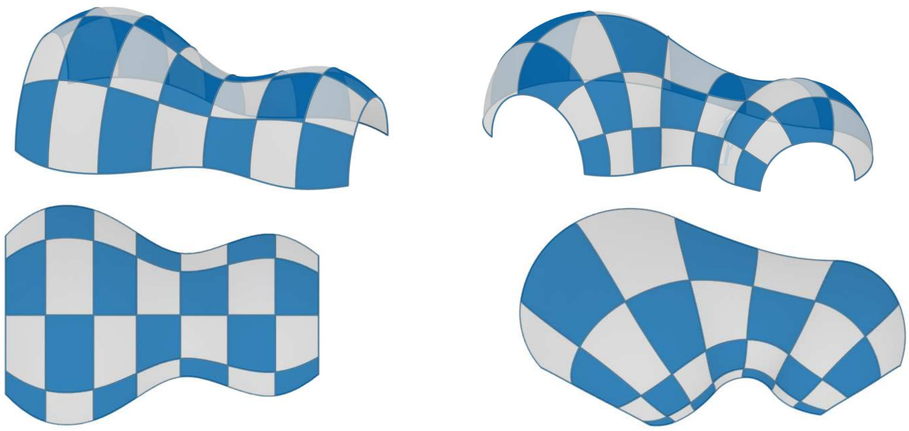

Figure 4: Surface of revolution (left) and one image of inversion (right)

The computation of the Möbius transform is based on Equation (1), it requires no optimization or matrix manipulation and can therefore be done as quickly as other simple transformations, like translations. The cyclidic net laying on the circular mesh is also generated in real-time. Möbius transforms can be applied on more complex shapes than surfaces of revolution, for example on Monge's surfaces or moulding surfaces (Mesnil *et al.* [15,16]).

4. Generalized cyclidic nets: a geometric approach for doubly curved crease

4.1. Doubly curved crease

Creases, understood as normal vector discontinuity, are an essential feature of free-form architecture. As an example, they are a well-known feature of Frank Gehry's architectural language and were also used by master designers like Eduardo Torroja for the 'Zarzuela' Hippodrome or Nicolas Esquillan in several designs (Billington [3], Marrey [14] Motro [17]). However, their construction remains a challenge, as creases are generally not aligned with lines of curvature, excluding the possibility to build them as conical meshes. Nicolas Esquillan's work gives good example of creased shells, but each solution was tailored for a specific project, leaving no general method to generate constructible creases (Motro [16]).

The usual modeling tools, even based on post rationalization of geometry hardly deal with the problem of discontinuity of normal vector in the rationalization of free-form structures. The only examples dealing with curved crease in architecture only consider developable surfaces (Kilian *et al.* [11]), which are sensitive to local buckling due to zero Gaussian curvature.

4.2. Generalized cyclidic net

In this paper, the authors generalize the construction rules of Cyclidic Nets to deal with discontinuities of normal vectors. The fundamental shapes remain cyclidic patches, the only difference with the usual cyclidic nets as defined in (Huhnen-Venedey and Bobenko [9]) is the reflection rule for the normal vectors, as illustrated in Figure 5**.**

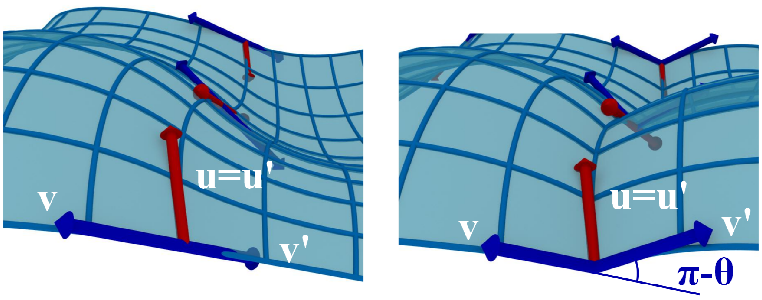

Figure 5: Reflection rule for cyclidic net (left) and for generalized cyclidic net (right)

When propagating the frame (u,v) of the cyclidic patch from one face to the other, the classical approach keeps one vector and inverts the other:

$$\begin{cases} u' = u \\ v' = -v \end{cases} \tag{3}$$

In (3), u refers to the common edge between the two patches. The first equality means that two patches have the same boundary; the second equality translates the continuity of tangent vector (C^1 surface). Therefore the second equality is not necessary if one only deals with C^0 surfaces. Equation (3) can thus be turned in its more general form:

$$\begin{cases} u' = u \\ v' = R_{u,\theta}.V \end{cases} \tag{4}$$

$R_{u,\theta}$ is a matrix representing the rotation along the vector u with an angle θ called *crease angle* in this paper. Equation (2) refers to an angle $\theta = \pi$. This generalized definition has to be coherent for any patch, which implies that the crease angle has to be constant along a polylines. Another restriction is that it is not possible to introduce creases in two different directions. A simple example with four patches is displayed in Figure *6*.

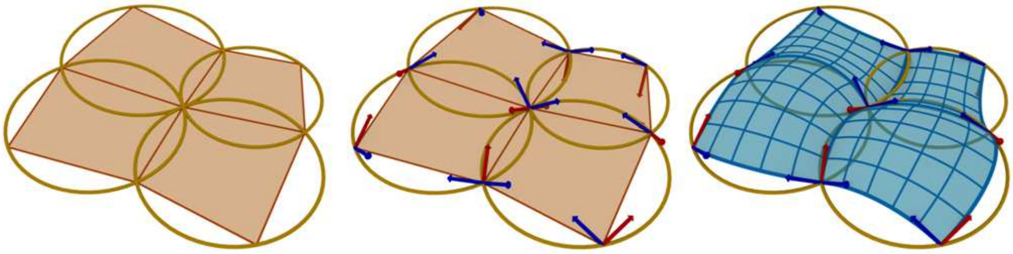

Figure 6: Circular Mesh and a Generalized Cyclidic Net

4.3. Application of curved creased

The introduction of Generalized Cyclidic Nets allows the drawing of constructible creases: the designer can control the crease angle and the number of creases. These parameters enrich considerably the design space, from regular creases which have a high structural potential, like the one presented in Section 5, to more aesthetical designs, like the one displayed in Figure 7. Note also that the Sydney Opera House geometry is a collection of spheres and can be seen as a special case of generalized cyclidic net.

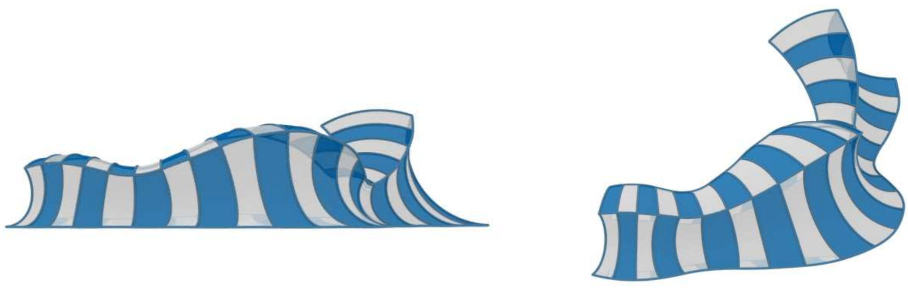

Figure 7: Doubly curved shape with a unique crease as Generalized Cyclidic Net

Any circular mesh can support a Generalized Cyclidic Net, therefore all the shapes presented as basis for circular meshes can be used to support Generalized Cyclidic Nets. Like cyclidic nets, these objects are natural objects of Möbius Geometry.

5. Application to structural design

5.1. Benefits of cyclidic nets for structural performance

Cyclidic nets have several advantages when structural design is concerned. Firstly, they allow for the construction of torsion-free-nodes, which are compatible with double-layer structures or arbitrary members' height: the designers can thus play freely with the structural height regardless of eventual geometrical difficulty. Secondly, the fabrication of curved members is made easy due to the fact that isoparametric lines of cyclidic patches are circles: complex structural hierarchies with a combination of straight secondary members and curved primary members can thus arise from cyclidic nets. Finally, unlike modeling techniques based on mesh optimization, they allow a quick remeshing, allowing the designers to modify the density of a structural layout in real time. This framework thus offers the possibility to explore efficiently the design space for structural engineers, from structural systems to structural density.

5.2. Crease and structural performance

General Cyclidic Nets offer the possibility to model constructible creases. Folding or creases play an important role in structural design, since they can stiffen a structure or increase its buckling load. Think of the famous example of a flat sheet of paper which can barely support its own weight whereas a single crease makes it significantly stiffer. In applications to doubly curved systems, crease can allow the designer to increase locally the Gaussian curvature of a shell by orders of magnitude. This idea is illustrated by a case study on Nicolas Esquillan's work.

5.3. Revisiting Esquillan's work

A perfect illustration of the structural potential of creases in doubly curved structures is the work of French Engineer Nicolas Esquillan. Some of his crease-shells designs held record of span at the time of their construction. This section focuses on the Marignane hangars, an early example of creased shell for Esquillan.

Designed in 1949 by Nicolas Esquillan and built in 1952, the Marignane hangars are a juxtaposition of torus-shaped shells spanning 101.5m for a rise of 12.1m (Motro [17]). At that time, it was the span record worldwide for a concrete shell. This record was later broken by the CNIT, another example of doubly-curved creased shell (Marrey [14], Motro [17]). The geometry of the Marignane hangars is shown on *Figure 8*. Previous examples of such geometry can also be found in the work of Franz Dischinger (Billington [3]).

The shape imagined by Nicolas Esquillan can easily be reinterpreted with generalized cyclidic nets. Tori are indeed a trivial subclass of Dupin cyclide. The reference Circular Mesh is a simple rectangular Mesh, such as the one displayed in Figure 8. Generalized Cyclidic Meshes give therefore a theoretical Framework that describes the geometry of Marignane hangars. The simplicity of the base circular mesh hints that the shape exploration of curved crease is still in its infancy. Generalized Cyclidic Nets should open new possibilities in this field.

Figure 8: Geometry of the Marignane Hangar: perspective (left), and side view (center)

6. Conclusion

Free-form structures are still a challenge for engineers in terms of fabrication and structural performance. Engineers and architects need interactive tools to assess the constructability of their designs. Both NURBS modeling and mesh optimization strategy have taken an increasing importance in complex structures, but they show some limitations. In this paper, the authors have introduced a method spanning between these two solutions. Generalized Cyclidic Meshes use a base circular mesh and generate complex shapes as a collection of NURBS parameterized by their lines of curvature, yielding automatically geometrically optimal structural layouts. This framework for fabrication-aware design also introduces a systematic way to generate constructible crease in free-form architecture, which allowed the authors to reinterpret the work of master builder Nicolas Esquillan. It creates new shapes thanks to inversion, and the potential offered by curved crease in structural optimization seems very promising.

Future work includes the development of a user friendly environment for the design of creased shells, exploration of the design space offered by creased shells and automatic methods for the generation of circular meshes. Innovative methods of structural optimization combining form-finding and creases can also arise from this work.

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References

[1] Adam P., Sur les surfaces isothermiques à lignes de courbure planes dans un système ou dans les deux systèmes, *Annales scientifiques de l'E.N.S.*, Gauthier-Villars, 1893 ; **10** ; 319-358.

[2] Bagneris M., Motro R., Maurin B. and Pauli N., Structural Morphology Issues in Conceptual Design of Double Curved Systems. *International Journal of Space Structures*, 2008; **23**; 79-87.

[3] Billington D. P., *The Tower and the Bridge: The New Art of Structural Engineering*, Princeton University Press, 1985.

[4] Bo P., Pottmann H., Killian M., Wang W., Wallner J., Circular Arc Structures, in *ACM Transactions on Graphics*, 2011; **30**(4):101-111.

[5] Deng B., Bouaziz S., Deus M., Kaspar A., Schwartzburg Y., Pauly M., Interactive Design exploration for constrained meshes, *Computer Aided Design*, 2015; **61**;13-23.

[6] Foufou, S., Garnier, L., and Pratt, M. J., Conversion of Dupin cyclide patches into rational biquadratic bézier form. In *Mathematics of Surfaces XI.* Springer Berlin Heidelberg, 2005 ; 201-218.

[7] Garnier L., Foufou, S., & Neveu, M. , Conversion d'un carreau de Bézier rationnel biquadratique en un carreau de cyclide de Dupin quartique. *TSI. Technique et science informatiques*, 2006 ; ***25***(6) ; 709-734.

[8] Glymph J., Shelden D., Ceccato C., Mussel J., Schober H., A parametric strategy for free-form glass structures using quadrilateral planar facets, *Automation in Construction*, 2004;**13**: 187-202.

[9] Huhnen-Venedey E., Bobenko A., Curvature line parametrized surfaces and orthogonal coordinate systems–Discretization with Dupin cyclides, *ArXiv preprint*, 2011, 1101.5955.

[10] Joshi, P., and Séquin, C., Energy minimizers for curvature-based surface functionals. *Computer-Aided Design and Applications*, 2007; ***4***(5): 607-617.

[11] Kilian, M., Flöry, S., Chen, Z., Mitra, N. J., Sheffer, A., & Pottmann, H., Developable surfaces with curved creases. *Proc. AAG*, 2008; 33-36.

[12] Krasauskas R., Circular Meshes, cyclide splines and geometric algebra, in *Algebraic geometry in the sciences*, Oslo, (2011).

[13] Liu Y., Pottmann H., Wallner J., Yang Y.-L., Wang W., Geometric Modeling with Conical Meshes and Developable Surfaces, in ACM Transactions on Graphics, 2006; **25**: 681-689.

[14] Marrey B., *Nicolas Esquillan: Un ingénieur d'entreprise*, Picard, 1992.

[15] Mesnil R., Douthe C., Baverel O., Léger B., Caron J.-F., Isogonal moulding surfaces: a family of shapes for high node congruence in free-form structures, Automation in Construction, 2015 (submitted).

[16] Mesnil R., Santerre Y., Douthe C., Baverel O., Léger B., Generating high node congruence in freeform structures with Monge's Surfaces, in *IASS 2015 Symposium Future Visions* (submitted)

[17] Motro R., Maurin B.. Bernard Laffaille, Nicolas Esquillan, Two French Pioneers. in *IASS-IABSE Symposium: Taller, Longer, Lighter*, 2011.

[18] Pottmann H., Liu Y., Wallner J., Bobenko A. and Wang W., Geometry of Multi-layer support structures for architecture, in *ACM Transactions on Graphics*, 2007; **26**(3).

[19] Pottmann H., Wallner J., The focal geometry of circular and conical meshes, *Advances in Computational Mathematics*, 2006; **29**: 249-268.

[20] Sheck H.J, The force density method for form-finding and computation of networks, *Computer methods in applied Mechanics and Engineering*, 1974; **3**; 115–134.

[21] Wang W., Liu Y.-L., Yan D., Chan B., Ling R. and Sun F. Hexagonal Meshes with Planar Faces, *Dept. of CS, HKU, Tech. Rep*, 2008; HKU CS Tech Report TR-2008-13.

[22] Yang Y.-L., Yang Y.-J., Pottmann H. and Mitra, N. J., Shape Space Exploration of Constrained Meshes in *ACM Transactions on Graphics*, 2011; **30**;124-133.